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Projective limits and balanced topological groups ☆

Gerald Itzkowitz

Queens College, Department of Mathematics, Flushing, NY 11367, USA

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Abstract

The purpose of these notes is to provide background material for a Workshop in Topological Groups given at the Summer Topology Conference at the C.W. Post campus of Long Island University during the period August 4–7, 1999. We will be looking at the two concepts of 'balanced groups' and 'functionally balanced groups'. In 1992 a paper was published by the author in conjunction with S. Rothman, H. Strassberg, and T.S. Wu where it was shown that the two concepts are equivalent for locally compact groups.

A "balanced group" is one in which the left and right uniform structures are equivalent and a "functionally balanced group" is one on which the classes of left uniformly continuous bounded real valued functions and those that are right uniformly continuous coincide. In our paper we were able to use projective limits of Lie groups to establish the desired result. This endeavor together with a classical theorem of Graev suggested that this result might be extended to those groups that are projective limits of metric groups.

The notes consist of four sections:

(1) Projective limits of topological groups.
(2) Functionally balanced groups.
(3) Projective limits of groups.
(4) Postscript, the work of Protasov and Saryev and the state of the problem.

Section 1 is a quick introduction with proofs to the concept of a projective limit of topological groups. Section 2 introduces the concept of a functionally balanced group. It describes an important characterization of such groups due to Protasov and Saryev and an important corollary of this characterization. Section 3 has a complete proof of Graev's motivating theorem and obtains a number of theorems, which to my knowledge are new, concerning projective limits of metric groups. Finally, Section 4 contains an account of a miniconference held at C.W. Post after the workshop ended and includes a proof of the Protasov–Saryev characterization which has not been available previously in English and also a result of Protasov that reduces the whole problem of 'functional balance' versus 'balance' to the solution of a simply stated question.

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E-mail address: zev@forbin.qc.edu (G. Itzkowitz).

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0. Introduction

The purpose of these notes is to provide background material for a workshop consisting of three lectures. The purpose of these lectures is to give a quick introduction to the concept of projective limits on an inverse limit system of topological groups, and then to illustrate the usefulness of this concept to establish several new theorems that characterize when the concept of a 'balanced group' coincides with the concept of a 'functionally balanced group'. These notes will be more complete than the lectures since it will include more proofs of important theorems than the lectures themselves. The concluding section of these notes includes a proof of a theorem of Protasov and Saryev that is crucial to our approach and a discussion based on another theorem of Protasov that shows that the solution of the question of 'balance' vs. 'functional balance' depends on the solution of an easily stated problem involving products of functionally balanced groups.

At this point we will loosely describe a balanced group as one where the right and left uniform structures are equivalent (give the same uniformity on $G \times G$). A functionally balanced group is one on which the classes of left and right uniformly continuous bounded real valued functions coincide. It is natural to wonder when the two concepts are equivalent. In fact, I have been involved with this problem since around 1970 when I showed that for locally compact metric groups the two concepts are equivalent [5]. More recently, starting in 1988, several researchers were able to extend these results to locally compact groups [3,8,11,12,15], almost metrizable groups [13], quasi-k groups [15], and locally connected groups [10]. The concept of a projective limit of a topological group has often proved to be very useful in my own research.

1. Projective limits of topological groups

In this section we basically follow the approach of Bourbaki [1]. For our purposes we start with a directed set A with partial ordering relation $\leqslant$. In the sequel, A will always denote a directed set when it is used as an index set. We will also assume that all topological groups are T_0 and therefore Hausdorff (except when described otherwise). We suppose that for each $\alpha \in A$ there is a topological group G_α and that for each pair (α, β) where $\alpha \leqslant \beta$ there is a homomorphism $f_{\beta\alpha}$ from G_β into G_α.

Definition. The set of ordered pairs $(G_\alpha, f_{\beta\alpha})$, $\alpha, \beta \in A$, $\alpha \leqslant \beta$, is an inverse limit system of topological groups if

(a) the $f_{\beta\alpha}$ are continuous homomorphisms of G_β into G_α if $\alpha \leqslant \beta$.
(b) If $\alpha \leqslant \beta \leqslant \gamma$ then $f_{\gamma\alpha} = f_{\beta\alpha} \circ f_{\gamma\beta}$.

Terminology. The maps $f_{\beta\alpha}$ will be called the connecting maps of the inverse limit system $(G_\alpha, f_{\beta\alpha})$, $\alpha, \beta \in A$, $\alpha \leqslant \beta$.

Definition. The projective limit $G = \operatorname{proj} G_\alpha$ of the inverse limit system $(G_\alpha, f_{\beta\alpha})$, $\alpha, \beta \in A$, $\alpha \leqslant \beta$, is the subset of $\prod_{\alpha\in A} G_\alpha$ consisting of all points $\{x_\alpha\}_{\alpha\in A}$ for which $x_\alpha = f_{\beta\alpha}\ (x_\beta)$ whenever $\alpha \leqslant \beta$.

Notation. If $G = \operatorname{proj} G_\alpha$ then the restriction of the projection map $\pi_\beta : \prod_{\alpha\in A} G_\alpha \to G_\beta$ to G is denoted by f_β.

Fact 1.1. *$f_\alpha(x) = f_{\beta\alpha}(f_\beta(x))$, if $\alpha \leqslant \beta$ so that $(f_\alpha)^{-1}(x) = (f_{\beta\alpha}(f_\beta(x)))^{-1} = (f_\beta)^{-1} \circ (f_{\beta\alpha})^{-1}(x)$.*

Here just note that $f_\alpha(x) = x_\alpha$ and $f_{\beta\alpha}(f_\beta(x)) = f_{\beta\alpha}(x_\beta) = x_\alpha$.

Definition. The topology on $G = \operatorname{proj} G_\alpha$ is the weakest topology making the maps $f_a : G \to G_\alpha$ continuous.

Remark 1.2. This definition is consistent with the definition of the product topology on $\prod_{\alpha\in A} G_\alpha$. In the case of the product topology a subbase for the open sets in $\prod_{\alpha\in A} G_\alpha$ consists of the sets of the form $(\pi_\alpha)^{-1}(U_\alpha)$, where U_α is open in G_α for $\alpha \in A$. This is the weakest topology making projections continuous. The base for the product topology consists of all finite intersections of such subbasic open sets. In the case of projective limits A is a directed set. Therefore if $\alpha(1), \alpha(2), \ldots, \alpha(n) \in A$ are given there is $\gamma \in A$ satisfying $\alpha(1) \leqslant \gamma, \alpha(2) \leqslant \gamma, \ldots, \alpha(n) \leqslant \gamma$. In this case as in the product topology case basic open sets in $\operatorname{proj} G_\alpha$ are of the form $\bigcap\{(f_{\alpha(i)})^{-1}(U_{\alpha(i)})\colon\ i = 1, 2, \ldots, n\}$, where $U_{\alpha(i)}$ is open in $G_{a(i)}$. Note that $(f_{\gamma\alpha(i)})^{-1}(U_{\alpha(i)}) = U_{\gamma,i}$ is open in G_γ, for $i = 1, 2, \ldots, n$. Therefore $U_\gamma = \bigcap\{U_{\gamma,i}\colon\ i = 1, 2, \ldots, n\}$ is open in G_γ. It now follows from Fact 1.1 that $(f_\gamma)^{-1}(U_\gamma) = \bigcap\{(f_{\alpha(i)})^{-1}(U_{\alpha(i)})\colon\ i = 1, 2, \ldots, n\}$.

Corollary 1.3. *The sets of the form $(f_\alpha)^{-1}(U_\alpha)$, where U_α is open in G_α, for $\alpha \in A$, is a base for the topology of G.*

Fact 1.4. *If each map $f_\beta : \operatorname{proj} G_\alpha \to G_\beta$ is onto then the connecting maps $f_{\gamma\beta} : G_\gamma \to G_\beta$ are onto.*

Proof. Let $x_\beta \in G_\beta$. Then there is $x \in \operatorname{proj} G_\alpha$ such that $f_\beta(x) = x_\beta$. Let $x_\gamma = f_\gamma(x)$. Now $x_\beta = f_\beta(x) = f_{\gamma\beta}(f_\gamma(x)) = f_{\gamma\beta}(x_\gamma)$. Therefore $f_{\gamma\beta}$ is onto. □

Theorem 1.5. *Let $(G_\alpha, f_{\beta\alpha})$, $\alpha, \beta \in A$, $\alpha \leqslant \beta$, be an inverse system of topological groups such that each natural homomorphism $f_\beta : \operatorname{proj} G_\alpha \to G_\beta$ is onto. If the connecting maps $f_{\beta\alpha}$ are open then the natural homomorphisms f_β are open maps.*

Proof. Let e be the identity in $\operatorname{proj} G_\alpha$ and let e_α be the identity in G_α, $\alpha \in A$. By Remark 1.2, each open neighborhood V of e contains a basic open set of the form $(f_\beta)^{-1}(V_\beta)$, where V_β is an open neighborhood of e_β in G_β. Now fix α. Since A is a directed set, there is $\gamma \in A$ satisfying $\alpha \leqslant \gamma$ and $\beta \leqslant \gamma$. Since the functions $f_{\beta\alpha}$ are continuous, $V_\gamma = (f_{\gamma\beta})^{-1}(V_\beta) \cap (f_{\gamma\alpha})^{-1}(V_\alpha)$ is open in $G\gamma$ and is a neighborhood of e_γ in G_γ. Furthermore,

$$\begin{aligned}(f_\gamma)^{-1}(V_\gamma) &= (f_\gamma)^{-1}\big((f_{\gamma\beta})^{-1}(V_\beta) \cap (f_{\gamma\alpha})^{-1}(V_\alpha)\big)\\ &= (f_\gamma)^{-1}\big[(f_{\gamma\beta})^{-1}(V_\beta)\big] \cap (f_\gamma)^{-1}\big[(f_{\gamma\alpha})^{-1}(V_\alpha)\big]\\ &= (f_\beta)^{-1}(V_\beta) \cap (f_\alpha)^{-1}(V_\alpha) \subset V,\end{aligned}$$

so that $V_\gamma = f_\gamma((f_\gamma)^{-1}(V_\gamma)) \subset f_\gamma(V)$. Therefore,

$$f_\alpha(V) = (f_{\gamma\alpha} \circ f_\gamma)(V) = f_{\gamma\alpha}\big[f_\gamma(V)\big] \supset f_{\gamma\alpha}(V_\gamma).$$

Since $f_{\gamma\alpha}$ is an open map of G_γ onto G_α, $f_{\gamma\alpha}(V_\gamma)$ is a neighborhood of e_α in G_α. Therefore $f_\alpha(V)$ is a neighborhood of e_α. Since this argument can be made at each point x in V it follows that $f_\alpha(V)$ is open in G_α, so that f_α is an open map. □

Notation 1.6. If A is a set in a topological space X then A^- will denote the closure of A in X and A^c will denote the complement of A in X.

In the following theorem we do not assume in advance that the inverse mapping system is made up of T_0 groups.

Theorem 1.7. *Let $(G_\alpha, f_{\beta\alpha})$, $\alpha, \beta \in A$, $\alpha \leqslant \beta$, be an inverse mapping system. Then $G = \operatorname{proj} G_\alpha$ is a subgroup of $\prod_{\alpha\in A} G_\alpha$. If all of the groups G_α are T_0 then G is T_0 and is a closed subgroup of $\prod_{\alpha\in A} G_\alpha$.*

Proof. It is an easy exercise that G is a group. Since T_0 groups are Hausdorff, then $\prod_{\alpha\in A} G_\alpha$ is Hausdorff and so G is Hausdorff (it is a subspace) and so T_0. Let now for $\alpha \leqslant \beta$,

$$F_{\beta\alpha} = \left\{x \in \prod_{\alpha\in A} G_\alpha\colon \pi_\alpha(x) = f_{\beta\alpha}(\pi_\beta(x))\right\}.$$

Then $\bigcap_{\alpha\leqslant\beta} F_{\beta\alpha} = G$. Furthermore each $F_{\beta\alpha}$ is closed. To see this, note that the functions π_α, π_β, and $f_{\beta\alpha}$, are continuous. Thus if $z \in (F_{\beta\alpha})^-$ there is a net $\{z_\gamma\}$ in $F_{\beta\alpha}$ converging to z. Thus $\pi_\alpha(z_\gamma) = f_{\beta\alpha}(\pi_\beta(z_\gamma))$ for each γ. Finally, by continuity

$$\pi_\alpha(z) = \lim_\gamma \pi_\alpha(z_\gamma) = \lim_\gamma f_{\beta\alpha}\big(\pi_\beta(z_\gamma)\big) = f_{\beta\alpha}\big(\pi_\beta(z)\big).$$

Thus $z \in F_{\beta\alpha}$ and $F_{\beta\alpha}$ is closed. G is closed since it is an intersection of closed sets. □

Note 1.8. When considering projective limits of Hausdorff spaces X_α the same proof can be used to show that the projective limit is a closed subspace of the product Hausdorff space $\prod_{\alpha\in A} X_\alpha$. From this point on our groups will again be T_0.

Definition 1.9. Let $\{x_\alpha\}_{\alpha\in B}$ be a net in a T_0 group G. Then $\{x_\alpha\}_{\alpha\in B}$ is said to be a Cauchy net with respect to the right uniform structure on G if for each symmetric neighborhood U of the identity there is a $\gamma \in B$ such that if $\alpha, \beta \geqslant \gamma$ then $x_\alpha(x_\beta)^{-1} \in U$. A discussion of the left and right uniform structure on G appears in Sections 4.2 and 4.3. We will call such a net a right Cauchy net. (A similar definition holds for a left Cauchy net. Simply replace $x_\alpha(x_\beta)^{-1} \in U$ by $(x_\beta)^{-1}x_\alpha \in U$ in the definition.)

Note that it is not hard to show that if a net in a topological group converges then it is both a left and right Cauchy net. Since inversion is continuous it is an easy exercise to show that every right Cauchy net converges in G iff every left Cauchy net converges.

Definition 1.10. The T_0 group G is complete if every right (or left) Cauchy net in G converges to a point of G.

Note 1.11. It is a standard fact for completely regular (uniform) spaces that a closed subspace of a complete space is itself complete. (If X is complete every Cauchy net in X converges. If Y is a closed subspace then a Cauchy net in Y is also a Cauchy net in X so it converges to a point x of X. However Y is closed so x is in Y.) It is also a fact that a product of complete uniform spaces is complete. Thus an application of Theorem 1.7 yields the following theorem.

Theorem 1.12.
(a) *Let G_α, $\alpha \in A$, be a collection of complete groups. Then $\prod_{\alpha\in A} G_\alpha$ is complete.*
(b) *If in addition the G_α, $\alpha \in A$, are an inverse limit system, then $G = \operatorname{proj} G_\alpha$ is complete.*

Fact 1.13. *If G is a T_0 group and H is a closed normal subgroup then G/H is T_0* [4, 5.21].

Theorem 1.14. *Let A be a directed set and let H_α, $\alpha \in A$, be a family of closed normal subgroups of the T_0 group G satisfying $H_\beta \subset H_\alpha$ whenever $\alpha \leqslant \beta$. Let $G_\alpha = G/H_\alpha$, for each $\alpha \in A$. Then the family $\{G_\alpha\colon \alpha \in A\}$ is an inverse limit system in which the connecting maps $g_{\beta\alpha}: G_\beta \to G_\alpha$, $\alpha \leqslant \beta$, are open and onto. Thus the system $(G_\alpha, g_{\beta\alpha})$, $\alpha, \beta \in A$, $\alpha \leqslant \beta$, has a projective limit $\mathcal{G} = \operatorname{proj} G_\alpha$.*

Proof. For each α, let $g_\alpha : G \to G_\alpha$ be the quotient map so that g_α is open, onto and continuous [4, Theorem 5.17]. Then for each $X \subset G$, $g_\alpha(X) = XH_\alpha$, for each $\alpha \in A$, and for $\alpha \leqslant \beta$, $XH_\beta \subset XH_\alpha$. Furthermore, if $g_\alpha(X) = XH_\alpha = X'$, then $g_\alpha(XH_\beta) = g_\alpha(XH_\beta H_\alpha) = g_\alpha(XH_\alpha) = X'$. Using this calculation, define $g_{\beta\alpha}$ by

$$g_{\beta\alpha}(XH_\beta) = g_\alpha(X) = XH_\alpha, \quad \text{for } X \subset G.$$

Note that $g_{\beta\alpha}(XH_\beta) = g_{\beta\alpha}(g_\beta(X)) = XH_\alpha = g_\alpha(X)$. Therefore $g_\alpha = g_{\beta\alpha} \circ g_\beta$ and $g_{\beta\alpha}: G_\beta \to G_\alpha$.

(1) $g_{\beta\alpha}$ *is a homomorphism*: Let $x', y' \in G_\beta$, then there are points $x, y \in G$ such that $g_\beta(x) = x'$ and $g_\beta(y) = y'$. Therefore,

$$\begin{aligned} g_{\beta\alpha}(x'y') &= (g_{\beta\alpha} \circ g_\beta)(xy) = g_\alpha(xy) = g_\alpha(x)g_\alpha(y) \\ &= g_{\beta\alpha} \circ g_\beta(x) g_{\beta\alpha} \circ g_\beta(y) = g_{\beta\alpha}(x')g_{\beta\alpha}(y'), \end{aligned}$$

and similarly $g_{\beta\alpha}[(x')^{-1}] = [g_{\beta\alpha}(x')]^{-1}$.

(2) $g_{\beta\alpha}$ *is continuous*: Let U_α be open in G_α. Then $(g_\alpha)^{-1}(U_\alpha) = V_\alpha H_\alpha$ for some open set $V_\alpha \subset G$. Therefore $V_\alpha H_\alpha$ is open in G. Since g_β is open, $g_\beta(V_\alpha H_\alpha) = V_\alpha H_\alpha H_\beta = V_\alpha H_\alpha = U_\beta$ is open in G_β. Now $g_{\beta\alpha}(U_\beta) = g_\alpha(V_\alpha) = V_\alpha H_\alpha = U_\alpha$, so that $(g_{\beta\alpha})^{-1}(U_\alpha) = U_\beta$, which is open in G_β.

(3) $g_{\beta\alpha}$ *is open*: Let U_β be open in G_β so that the coset $V_\beta H_\beta$ in G satisfying $g_\beta(V_\beta H_\beta) = U_\beta$, is open. From the definition of $g_{\beta\alpha}$,

$$g_{\beta\alpha}(U_\beta) = g_\alpha(V_\beta H_\beta) = V_\beta H_\beta H_\alpha = V_\alpha H_\alpha = U_\alpha,$$

is open in G_α, since g_α is an open map.

This shows that the family $(G_\alpha, g_{\beta\alpha})$, $\alpha, \beta \in A$, $\alpha \leqslant \beta$, is an inverse limit system so that $\mathcal{G} = \operatorname{proj} G_\alpha$ exists. □

Notes 1.15.

(1) Theorem 1.5 and the method of proof showing that the maps $g_{\beta\alpha}$ are open imply that more generally with projective limits, if the maps g_α, $\alpha \in A$, are onto then they are open iff all the connecting maps $g_{\beta\alpha}$ are open.

(2) It is now natural to ask if the two groups, G and $\mathcal{G}$ coincide. Our next theorem shows that G is a dense subgroup of $\mathcal{G}$ and that if even one of the H_α, $\alpha \in A$, is complete then $G = \mathcal{G}$.

(3) The elements of $\mathcal{G}$ are of the form $x' = \{x_\alpha\}_{\alpha \in A} = \{g_\alpha(x)\}_{\alpha \in A}$, where $x \in G$. Let $g = \{g_\alpha\}_{\alpha \in A}$ where $g(x) = \{g_\alpha(x)\}_{\alpha \in A}$. The point $x \in xH_\alpha = (g_\alpha)^{-1}[g_\alpha(x)]$, so that $x \in \bigcap_{\alpha \in A} xH_\alpha$. Furthermore, if $y \in \bigcap_{\alpha \in A} xH_\alpha \subset G$, then for each α: $y \in xH_\alpha$ so that $g_\alpha(y) = xH_\alpha \in G_a$. Therefore $g(y) = \{g_\alpha(x)\}_{\alpha \in A} \in \mathcal{G}$, and $y \in (g_\alpha)^{-1}[g_\alpha(x)]$. Conversely, if $y \in (g_\alpha)^{-1}[g_\alpha(x)]$, for each $\alpha \in A$, then $y \in \bigcap_{\alpha \in A} (g_\alpha)^{-1}[g_\alpha(x)] = \bigcap_{\alpha \in A} xH_\alpha$. Thus we get the following corollaries.

Corollary 1.16. *The kernel of the map* $g : G \to \mathcal{G}$ *is* $\bigcap_{\alpha \in A} H_\alpha$.

Corollary 1.17. *If* $\{e\} = \bigcap_{\alpha \in A} H_\alpha$, *then* g *is* 1–1 *from* G *into* $\mathcal{G}$.

Theorem 1.18. *Let* A *be a directed set and let* H_α, $\alpha \in A$, *be a family of closed normal subgroups of the* T_0 *group* G *that satisfy*:

(a) *Each neighborhood of* e *in* G *contains at least one* H_α.

(b) $H_\beta \subset H_\alpha$ *whenever* $\alpha \leqslant \beta$.

Then the map $g : G \to \mathcal{G} = \operatorname{proj} G/H_\alpha$ *is a* 1–1 *continuous open homomorphism onto a dense subgroup of the* T_0 *group* $\mathcal{G}$. *If in addition one of the* H_α *is complete* (*or compact*) *then* g *is onto.*

Proof. Let $G_\alpha = G/H_\alpha$, $\alpha \in A$. Then the canonical map $g_\alpha : G \to G_a$ is continuous, onto, and open for each $\alpha \in A$ [4, 5.17]. Since each H_α is closed, each G_α is T_0. Thus as before $\mathcal{G}$ is T_0. Furthermore, by Theorem 1.13 the canonical maps $g_{\beta\alpha} : G_\beta \to G_\alpha$ are continuous, onto, and open, so that the canonical maps $f_\alpha : \mathcal{G} \to G_\alpha$ are continuous, onto, and open. By Fact 1.16, the map g has kernel $\bigcap_{\alpha\in A} H_\alpha$, which is a closed subgroup of G. Since each neighborhood of e contains some H_α and since G is T_0 we have

$$\{e\} \subset \bigcap_{\alpha\in A} H_\alpha \subset \bigcap \bigl\{U\colon\ U \text{ a neighborhood of } e\bigr\} = \{e\},$$

so that $\bigcap_{\alpha\in A} H_\alpha = \{e\}$.Therefore g is 1–1 into.

(1) $g(G)$ *is dense in* $\mathcal{G}$: If U is open in $\mathcal{G}$ there is $\alpha \in A$ such that $(f_\alpha)^{-1}(U_\alpha) \subset U$, where U_α is open in G_α. Therefore $[(g^{-1}) \circ (f_\alpha)^{-1}](U_\alpha) \subset g(U)$. However the canonical map $g_\alpha : G \to G_\alpha$ is onto and $g_\alpha = f_\alpha \circ g$. Therefore $[(g^{-1}) \circ (f_\alpha)^{-1}](U_\alpha) = (f_\alpha \circ g)^{-1}(U_\alpha) = (g_\alpha)^{-1}(U_\alpha)$ is nonempty. This shows that $g^{-1}(U) \neq \emptyset$ and so $g(G) \cap U \neq \emptyset$.

(2) g *is open from* G *onto* $g(G)$: Let V be a neighborhood of e in G. Then there is a neighborhood W of e, such that $W^2 \subset V$ and an index $\alpha \in A$ such that $H_\alpha \subset W$. Then $WH_\alpha \subset W^2 \subset V$. Furthermore, $WH_\alpha = (g_\alpha)^{-1}(g_\alpha(W)) = (g^{-1})[(f_\alpha)^{-1}(g_\alpha(W))]$. Since $(f_\alpha)^{-1}(g_\alpha(W))$ is a neighborhood of the identity in $\mathcal{G}$, so is $g(WH_\alpha)$.

(3) g *is onto if one* H_γ *is complete*: Let $\{x_\alpha\}_{\alpha\in A} \in \mathcal{G}$. Then $(g_\alpha)^{-1}(x_\alpha) = \bar{x}_\alpha H_\alpha$, for some $\bar{x}_\alpha \in G$ and $\alpha \in A$. To prove that g is onto we need to show that $\bigcap_{\alpha\in A}(g_\alpha)^{-1}(x_\alpha) = \bigcap_{\alpha\in A} \bar{x}_\alpha H_\alpha \neq \emptyset$. [If $y \in \bigcap_{\alpha\in A} \bar{x}_\alpha H_\alpha$ then $g_\alpha(y) = x_\alpha$, for each $\alpha \in A$, so that $g(y) = \{x_\alpha\}_{\alpha\in A} \in \mathcal{G}$.]

Since H_γ is complete so is the translate $\bar{x}_\gamma H_\gamma$, since translates are homeomorphisms. Let U be a neighborhood of e in G. Then U contains an H_δ. Since A is directed there is a $\beta \in A$ such that $\gamma \leqslant \beta$ and $\delta \leqslant \beta$. Therefore $H_\beta \subset H_\gamma \cap H_\delta$. Therefore, $\bar{x}_\gamma H_\beta \subset \bar{x}_\gamma H_\delta$ $(\subset \bar{x}_\gamma U)$. Furthermore, $x_\gamma = g_\gamma(\bar{x}_\gamma) = g_{\beta\gamma}(g_\beta(\bar{x}_\gamma))$ so that $g_\beta(\bar{x}_\gamma) \in (g_{\beta\gamma})^{-1}(x_\gamma)$. Since $g_{\beta\gamma}(x_\beta) = x_\gamma$, we have $x_\beta \in (g_{\beta\gamma})^{-1}(x_\gamma)$ so that

$$\begin{aligned}\bar{x}_\beta H_\beta &= (g_\beta)^{-1}(x_\beta) \subset (g_\beta)^{-1}\bigl[(g_{\beta\gamma})^{-1}(x_\gamma)\bigr]\\ &= (g_{\beta\gamma} \circ g_\beta)^{-1}(x_\gamma) = (g_\gamma)^{-1}(x_\gamma) = \bar{x}_\gamma H_\gamma.\end{aligned}$$

Therefore for each $\gamma \leqslant \beta$, we have $\bar{x}_\beta H_\beta \subset \bar{x}_\gamma H_\gamma$. Since the subgroups $\{H_\alpha\colon\ \alpha \in A\}$ are closed in G it follows that the coset $\bar{x}_\beta H_\beta$ is complete. Furthermore, this argument can be used to show that $\bar{x}_\beta H_\beta \subset \bar{x}_\delta H_\delta$. We now observe that if $x, y \in \bar{x}_\beta H_\beta$ then there are $h_1, h_2 \in H_\beta$ such that $x = \bar{x}_\beta h_1$ and $y = \bar{x}_\beta h_2$ so that

$$x^{-1}y = (\bar{x}_\beta h_1)^{-1}\bar{x}_\beta h_2 = (h_1)^{-1}(\bar{x}_\beta)^{-1}\bar{x}_\beta h_2 = (h_1)^{-1}h_2 \in H_\beta \subset H_\delta \subset U.$$

The above argument shows that if $\beta \leqslant \omega$ then $\bar{x}_\omega H_\omega \subset \bar{x}_\beta H_\beta$. Since U is an arbitrary neighborhood of e it follows that $\{\bar{x}_\alpha\}_{\gamma\leqslant\alpha}$ is a Cauchy net on $\bar{x}_\gamma H_\gamma$ and so it converges to a point z of $\bar{x}_\gamma H_\gamma$. Thus $z \in \bar{x}_\beta H_\beta$ for all $\gamma \leqslant \beta$. Since $\bar{x}_\beta H_\beta \subset \bar{x}_\alpha H_\alpha$ for all $\alpha \leqslant \beta$, it follows that $z \in \bigcap_{\alpha\in A} xH_\alpha$. Therefore $g(z) = \{x_\alpha\}_{\alpha\in A} \in \mathcal{G}$ and g is onto. $\square$

Corollary 1.19. *If G is a σ-compact locally compact* (*or compact*) *T_0 group then G is a projected limit of σ-compact locally compact* (*or compact*) *T_0 metric groups.*

Proof. First note that $G = \bigcup_{i\in\mathbb{N}} K_i$, where each K_i is compact. It is an elementary fact that if F is compact and if V is a neighborhood of e then there is a symmetric neighborhood W of e such that $W \subset xVx^{-1}$ for all $x \in F$. This means that $W \subset \bigcap_{x\in F} xVx^{-1} = x(\bigcap_{x\in F} V)x^{-1}$.

Let U be a symmetric neighborhood of e in G with compact closure. Inductively, let $V_0 = U$ and select sequences $V_n \supset W_n$, of symmetric neighborhoods of e such that $(V_n)^2 \subset V_{n-1}$, $\bigcap_{i\leqslant n} xV_ix^{-1} = x(\bigcap_{i\leqslant n} V_i)x^{-1} \supset W_n$ for all $x \in F(n)$, where W_n is a neighborhood of e, and $F(n) = \bigcup_{i\leqslant n} K_i$. This means that $W_n \subset \bigcap_{x\in F(n)}\{\bigcap_{i\leqslant n} xV_nx^{-1}\}$. It is now an easy exercise to show that $H = \bigcap_{i\in\mathbb{N}} W_i$ is a compact normal subgroup of G. Clearly $H \subset U$ and H is a compact G_δ. This verifies that G satisfies condition (a) of the previous theorem. Condition (b) is easy to check so that Theorem 1.18 applies. It is clear that G/H is σ-compact and locally compact. Since H is a compact G_δ, it follows that $\{e_{G/H}\}$ is a countable intersection of open sets and therefore G/H is metrizable [4, Theorem 8.5]. A slightly modified and similar proof will confirm the compact case. □

2. Functionally balanced groups

Definition. A real valued function f on a topological group G is *left uniformly continuous* if for each $\varepsilon > 0$, there is a neighborhood W of e such that $|f(x) - f(y)| < \varepsilon$ whenever $x^{-1}y \in W$. f is *right uniformly continuous* if for each $\varepsilon > 0$, there is a neighborhood V of e such that $|f(x) - f(y)| < \varepsilon$ whenever $yx^{-1} \in V$.

Note 2.1. The condition for left uniform continuity is equivalent to the statement: for each $\varepsilon > 0$, there is a neighborhood W of e such that $|f(x) - f(y)| < \varepsilon$ whenever $y \in xW$. The statement for right uniform continuity is equivalent to the statement: for each $\varepsilon > 0$, there is a neighborhood V of e such that $|f(x) - f(y)| < \varepsilon$ whenever $y \in Vx$. In uniform spaces [9, p. 180] the more general definition appears: "If $f : X \to Y$, where $(X, \mathcal{U})$, $(Y, \mathcal{V})$ are uniform spaces, then f is uniformly continuous relative to $\mathcal{U}$ and $\mathcal{V}$ iff for each $V \in \mathcal{V}$ the set $U = \{(x, y)\colon (f(x), f(y)) \in V\}$ is a member of $\mathcal{U}$. In the topological group situation we start with a base $\mathcal{U}$ of the neighborhood system at e in G. Let $W \in \mathcal{U}$ and let $\overline{W} = \{(x, y)\colon x^{-1}y \in W\}$. Then $\overline{W}$ is an entourage of the diagonal Δ in $G \times G$. Similarly, $\overline{V} = \{(a, b)\colon |a - b| < \varepsilon\}$ is an entourage of the diagonal Δ in $\mathbb{R} \times \mathbb{R}$. If we let $f_2(x, y) = (f(x), f(y))$ we have the statement: f is left uniformly continuous iff $(f_2)^{-1}(\overline{V}) \supset \overline{W}$. Therefore $(f_2)^{-1}(\overline{V})$ is an entourage of Δ in $G \times G$.

Remark 2.2. My research in uniform spaces was originally motivated by the realization that the $U[x] = \{y\colon (x, y) \in \overline{U}\}$ of a uniform space becomes xU when considering the left uniformity and Ux when considering the right uniformity on a topological group G. This discovery had been motivated by a question asked by Nadler. He had asked me first to prove a theorem on topological groups similar to one that he and Frazer had proven to be true for metric spaces. This turned out to be fairly straight forward. Afterward, he asked me to generalize the result to uniform spaces. Once I realized the above identifications of the notation used on uniform spaces with the notation used on topological groups, I was able to prove several similar type theorems in the case of uniform spaces [6]. This identification led to much of my subsequent investigations on uniform spaces and on the left and right uniformities on a topological group [5,7,8].

Definition. A topological group G is balanced if the left and right uniformities are equivalent on G.

Note 2.3. This is equivalent to saying that given a neighborhood U of e in G there is a neighborhood V of e such that $xV \subset Ux$ for all x in G. This last condition says that $V \subset x^{-1}Ux$, for all x in G. Thus $V \subset \bigcap_{x\in G} x^{-1}Ux$ and so $W = \bigcap_{x\in G} x^{-1}Ux$ is a neighborhood of e. Now observe that $W \subset U$, and $y^{-1}Wy = y^{-1}(\bigcap_{x\in G} x^{-1}Ux)y = \bigcap_{x\in G} y^{-1}x^{-1}Uxy = \bigcap_{x\in G}(xy)^{-1}Uxy = \bigcap_{z\in G}(z)^{-1}Uz = W$, for all y in G.

Definition. Symmetric neighborhoods satisfying this last condition $W = x^{-1}Wx$, for all x in G, are called balanced or invariant neighborhoods. Thus we get:

Theorem 2.4. *A topological group G is balanced iff it has a neighborhood base at e consisting of balanced neighborhoods.*

Definition. A topological group G is functionally balanced if the class of left uniformly continuous bounded real valued functions on G coincides with the right uniformly continuous ones.

Note 2.5. In my early research (around 1970) I discovered that if G is a locally compact metric group then the conditions 'G is balanced' and 'G is functionally balanced' are equivalent [5]. One of the objects of our lectures is to show that the two conditions are equivalent if G is a projective limit of groups which are Lindelöff and of the second category in themselves. (This means that if G is a projective limit of complete Lindelöff metric groups then G is functionally balanced iff G is balanced.)

Protasov in a paper [13], written in 1991, where the problem is solved positively for the class of almost metrizable groups (which include locally compact and metric groups) refers to a result he and Saryev [14] published in 1988. The result gives a characterization of functionally balanced groups that is similar to a characterization of balanced groups appearing in Note 2.3. The statement in [13] is not complete. Therefore a complete statement will be given since it is of some interest and this theorem is crucial to these lectures. A proof of this theorem is given in Section 4. In [13], Protasov makes use of this characterization to prove that in the case of metric groups 'balance' and 'functional balance' are equivalent. Troallic, in [15], shows that this equivalence extends to quasi-k groups and gives an independent proof showing that the equivalence holds for metric groups.

Notation. Let $\mathcal{U}$ be the neighborhood system at e in the topological group G.

Theorem 2.6 (Protasov and Saryev). *Let G be a T_0 topological group. Then the following are equivalent*:

(a) *G is functionally balanced.*

(b) *For each $U \in \mathcal{U}$ and A in G there is a set $V \in \mathcal{U}$ such that $VA \subset AU$ and for each $W \in \mathcal{U}$ there is $V' \in \mathcal{U}$ such that $AV' \subset WA$.*

Definition. A topological group G is strongly functionally balanced if the class of all real valued left uniformly continuous functions coincides with the class of all right uniformly continuous functions on G.

Question 2.7. Does condition (b) imply that G is strongly functionally balanced? The motivation for this question comes from the proof of the Saryev–Protasov theorem that appears in Section 4.

Example 2.8. When considering uniform continuity on $\mathbb{R}$, the function $f(x) = x^2$ (or any polynomial of degree greater than 1) is not uniformly continuous on $\mathbb{R}$. However, $f(x)$ is uniformly continuous on any bounded interval in $\mathbb{R}$. Moreover each function $f_M(x) = f(x) \wedge M$ is uniformly continuous on $\mathbb{R}$. Thus a pointwise limit of uniformly continuous functions is not necessarily a uniformly continuous function. These considerations (and similar ones) hold on topological groups for right and left uniformly continuous real valued functions. It is a well-known folk theorem that the uniform limit of real valued uniformly continuous, left uniformly continuous, and right uniformly continuous functions is respectively uniformly continuous, left uniformly continuous and right uniformly continuous. However it is an easy exercise to show that uniform limits of such bounded functions are themselves bounded. This means that the additive groups of real valued bounded uniformly continuous, left uniformly continuous, and right uniformly continuous functions are all closed under the sup norm topology. Therefore we cannot conclude that (a) implies that G is strongly functionally bounded.

The theorem of Protasov and Saryev have as a corollary the following theorem.

Theorem 2.9. *If f is an open continuous homomorphism of G onto G' and if G is functionally balanced then G' is functionally balanced.*

Proof. Let $\mathcal{V}$ be the neighborhood system of the identity in G'. Let $U \in \mathcal{V}$ and let $A \subset G'$ be arbitrary. Note that $f^{-1}(AU) = f^{-1}(A)f^{-1}(U)$. Since G is functionally balanced there is an open neighborhood $V^{\sim}$ of the identity in G that satisfies $V^{\sim}f^{-1}(A) \subset f^{-1}(A)f^{-1}(U)$. [Note that $f^{-1}(U)$ is an open neighborhood of the identity in G because f is continuous.] Then

$$f\bigl(V^{\sim}f^{-1}(A)\bigr) = f(V^{\sim})f\bigl[f^{-1}(A)\bigr] = f(V^{\sim})A \subset f\bigl[f^{-1}(A)\bigr]f\bigl[f^{-1}(U)\bigr] = AU.$$

Since f is an open homomorphism it follows that $V = f(V^{\sim})$ is an open neighborhood of the identity in G'. Thus there is V in $\mathcal{V}$ for which $VA \subset AU$. The other set containment in Theorem 2.6(b) is proved similarly. Thus G' is functionally balanced. $\square$

3. Projective limits of groups

The motivation for the work in this section is a classical theorem of Graev [2] that appeared in 1950. The theorem is stated as Example 8.17 in [4] with a partial proof. Graev's theorem was not stated in terms of balance, but in terms of a neighborhood base at e consisting of sets U satisfying $xUx^{-1} = U$. It is also a folk theorem (see [4, 8.18]) for metric groups that the following are equivalent:

(1) G is balanced.

(2) G has a 2-sided invariant metric compatible with its topology.

(3) There is a countable neighborhood base $\{U_n\colon n < w\}$ of the identity consisting of balanced neighborhoods.

Graev's theorem can be stated as follows:

Theorem 3.2 (Graev). *Let G be a balanced T_0 group. Then G is topologically isomorphic with a subgroup of a product of balanced metric groups.*

Proof. Let $\mathcal{U}$ be a neighborhood base at e consisting of symmetric balanced sets. Note that if we fix U in $\mathcal{U}$ we can select a sequence of symmetric balanced neighborhoods U_n of e in $\mathcal{U}$ satisfying $U = U_0$ and $(U_n)^2 \subset U_{n-1}$, for n in $\mathbb{N}$. Since $xU_nx^{-1} = U_n$ for each n, we have that

$$x\left(\bigcap_{n\in\mathbb{N}} U_n\right)x^{-1} = \bigcap_{n\in\mathbb{N}}(xU_nx^{-1}) = \bigcap_{n\in\mathbb{N}} U_n.$$

It is a simple exercise to check that $H = \bigcap_{n\in\mathbb{N}} U_n$ is a closed normal subgroup of G. Therefore G/H is a T_0 group. This shows that each symmetric balanced neighborhood of e contains a closed normal G_δ subgroup. Let $f : G \to G/H$ be the natural homomorphism. Let $\tau(G/H)$ be the weakest topology on G/H making the collection $\{f(U_n)\colon n \in \mathbb{N}\}$ a neighborhood base at $e_{G/H}$. (Such a topology exists because the natural homomorphism f is continuous and open.) Note that $\tau(G/H)$ is metrizable because it has a countable base at the identity of G/H. Because H is normal, note that if x is in G/H and $x^\sim$ is in $f^{-1}(x)$ then

$$\begin{aligned} f^{-1}\big(xf(U_n)x^{-1}\big) &= f^{-1}(x)U_nHf^{-1}(x^{-1}) = f^{-1}(x)U_nf^{-1}(x^{-1})H \\ &= (x^\sim H)U_n\big((x^\sim)^{-1}H\big)H = (x^\sim)U_n(x^\sim)^{-1}H = U_nH. \end{aligned}$$

Therefore

$$xf(U_n)x^{-1} = f\big(f^{-1}(xf(U_n)x^{-1})\big) = f(U_nH) = f(U_n)$$

and we conclude that G/H has a countable neighborhood base at $e_{G/H}$ consisting of balanced symmetric neighborhoods. Therefore G/H with the topology $\tau(G/H)$ is a balanced metric group.

We showed that each balanced neighborhood U_α of e in G has a closed normal G_δ subgroup H_α and we note that the collection $\mathcal{U}$ of balanced neighborhoods of e is a directed set if we use $U \leqslant V$ iff $V \subset U$. This means we can use $\mathcal{U}$ to self index the balanced neighborhoods of e and the resultant closed normal G_δ subgroups of G. (I.e., for each U, H_U will be the corresponding subgroup.) When we refer to the directed index set we will use A instead of $\mathcal{U}$, U_α instead of U and H_α instead of H_U.

Let x be in G. For each $\beta \in A$ define $f_\beta : G \to G/H_\beta$ by $f_\beta(x) = xH_\beta$. If we regard the f_β as the normal quotient map onto G/H_β then f_β is continuous and therefore it is continuous onto G/H_β with the topology $\tau(G/H_\beta)$ on G/H_β since this topology is weaker than the quotient topology.

Let F be a closed set not containing e. Then F^c is a neighborhood of e, and so contains a symmetric balanced neighborhood U_α which in turn contains a closed normal G_δ subgroup H_α. It is clear that f_α separates e and F. Since translations are homeomorphisms there is always an f_α that separates a closed set from a point disjoint from it. Similarly since points are closed there is an f_β that separates points. Now define $F(x) = \{xH_a\}_{\alpha\in A} = \{f_\alpha(x)\}_{\alpha\in A} \in \prod_{\alpha\in A} G_a$ where $G_\alpha = G/H_a$, with the $\tau(G/H_\alpha)$ topology. Since the functions f_α, $\alpha \in A$, are continuous it follows from the Embedding Lemma [9, p. 116] that F is a homeomorphism of G onto $F(G) \subset \prod_{\alpha\in A} G_\alpha$. It is clear that F is a homomorphism and therefore a topological isomorphism, proving the theorem. □

Theorem 3.3 [4, 8.5]. *Let G be a locally countably compact group then G is metrizable iff $\{e\}$ is a countable intersection of open sets.*

Remark 3.4. In the proof it is shown that if $\{U_n\colon n < \omega\}$ is the family whose intersection is $\{e\}$, then the collection of all finite intersections of this family is a neighborhood base at e for the topology of G. If all the G/H_α of Theorem 3.2 are locally compact then the families of the form $\{f_\alpha(U_n)\colon n < \omega\}$ that we use in Theorem 3.2 generate the neighborhood bases in the respective G/H_α under the quotient topology. In this case each f_α is open and it can be shown as in Theorems 1.14 and 1.18 that G is topologically isomorphic with a dense subgroup of the projective limit $\mathcal{G}$ of the G/H_α. Since the G/H_α are locally compact they are complete and therefore $\mathcal{G}$ is complete. In the case that G itself is complete then the closed groups H_α are themselves complete. Thus in this case the function $g : G \to \mathcal{G}$ is onto. Thus G is in fact a projective limit of locally compact groups.

Theorem 3.5. *Let G be a balanced T_0 group.*

(i) *If G is locally compact then G is a projective limit of balanced locally compact metric groups.*

(ii) *If G is complete and if each G/H_α of Theorem* 3.2 *is locally compact in the quotient topology then G is a projective limit of locally compact metric groups.*

General Problem 3.6. Characterize those balanced groups that are projective limits of balanced metric groups.

As mentioned previously, Theorem 3.2 was the motivation for the work that comes next. When I first looked at the partial proof of Graev's Theorem in [4], I understood it to show that G is a projective limit of balanced metric groups. This of course was not the intent of the authors as became evident when I attempted to give a complete proof of the theorem. However, my original interpretation of the theorem suggested that if G is a projective limit of metric groups, then the concept of "balance" and the concept of "functionally balanced" are equivalent on G. As we shall see in the sequel this is the case when G is a projective limit of complete and Lindelöf metric groups.

Theorem 3.7. *Let G and H be T_0 topological groups such that G is Lindelöf and H is a Lindelöf group of the second category in itself. If $f : G \to H$ is a continuous surjective homomorphism then f is an open map.*

Proof. Since f is continuous, the graph of f is closed in $G \times H$ [9, p. 213]. Thus by [9, p. 213] we need to show that $[f(U)]^-$ is a neighborhood of the identity in H for each neighborhood U of the identity in G to conclude that f is open.

Let U be an open neighborhood of the identity in G. Since G is Lindelöf, G is covered by a countable number of translates of U. This means there is a sequence of translates $\{x_i U\colon i \in \mathbb{N}\}$ such that $G = \bigcup\{x_i U\colon i \in \mathbb{N}\}$. Therefore

$$H = f(G) \subset \bigcup \bigl\{f(x_i) f(U)\colon i \in \mathbb{N}\bigr\} \subset \bigcup \bigl\{f(x_i) f(U)^-\colon i \in \mathbb{N}\bigr\} \subset H.$$

Thus the closed sets $\{f(x_i) f(U)^-\colon i \in \mathbb{N}\}$ cover H and therefore one of the sets, say $f(x_j) f(U)^-$ contains an open set. Since translations are homeomorphisms $f(U)^-$ contains an open set.

Since U is an open neighborhood of the identity there is a symmetric neighborhood V of e_G such that $V^2 \subset U$. The above argument shows that $[f(V)]^-$ contains an open set. Furthermore,

$$\bigl[f(U)\bigr]^- \supset \bigl[f(V^2)\bigr]^- \supset \bigl[f(V)\bigr]^- \bigl[f(V)\bigr]^-.$$

Since V is symmetric so is $f(V)$, and

$$\bigl[f(V)\bigr]^- = \bigl[f(V^{-1})\bigr]^- = \bigl[f(V)^{-1}\bigr]^- = \bigl\{\bigl[f(V)\bigr]^-\bigr\}^{-1}.$$

By the Banach–Kuratowski–Pettis Theorem [9, p. 211], $[f(V)]^- \{[f(V)]^-\}^{-1}$ contains a neighborhood of e_H so that $[f(U)]^-$ is a neighborhood of e_H. Thus f is an open map. $\square$

Theorem 3.8. *Let $G = \prod_{\alpha \in A} G_a$, then G is balanced iff each G_α is balanced.*

Proof. Suppose G is balanced, then for each neighborhood U of e in G, there is a neighborhood V of e such that $x^\sim V \subset U x^\sim$ for all $x^\sim$ in G. Let U_α be a neighborhood of e_α in G_α. Then if π_α is the projection map of G onto G_α, $(\pi_\alpha)^{-1}(U_\alpha) = (U_\alpha)^\sim$ is open in G. Therefore there is a neighborhood $V^\sim$ of the identity in G satisfying $x^\sim V^\sim \subset (U_\alpha)^\sim x^\sim$ for all $x^\sim$ in G. If we let $V = \pi_\alpha(V^\sim)$ then since π_α is open it follows that $xV \subset U_\alpha x$ for each x in G_α, so that each G_α is balanced.

Suppose each G_α is balanced. Let U be an open neighborhood of e in G. Then without loss of generality we may assume that U is a basic open neighborhood of e. Therefore

$$U = \bigcap_{i \leqslant n} (\pi_{\alpha(i)})^{-1}(U_{\alpha(i)}) = \prod_{i \leqslant n} U_{\alpha(i)} \times \prod_{\alpha \notin \{\alpha(1),\ldots,\alpha(n)\}} G_\alpha.$$

Since each $G_{\alpha(i)}$, $i \leqslant n$, is balanced it follows that there are symmetric neighborhoods $V_1, \ldots, V_n$ of the identity in $G_{\alpha(1)}, \ldots, G_{\alpha(n)}$ respectively such that for each i, $x_{\alpha(i)} V_i \subset U_{\alpha(i)} x_{\alpha(i)}$ for all $x_{\alpha(i)}$ in $G_{\alpha(i)}$. It is now easy to check that if $V = \prod_{i \leqslant n} V_i \times \prod_{\alpha \notin \{\alpha(1), \ldots, \alpha(n)\}} G_\alpha$ then $xV \subset Ux$ for all x in G. $\square$

Note 3.9. In Bourbaki [1] after a long and complicated discussion of uniformities Theorem 3.8 is stated without proof. This gives the impression that an argument using the definition of product uniformities must be used in the proof. However, as we have seen the argument really is a simple one depending purely on the definition of the product topology on G. In view of this fact, the fact that the projections are continuous (and even right and left uniformly continuous), open, and surjective together with Theorem 2.9, allows us to conclude for T_0 groups:

Corollary 3.10. *Let $G = \prod_{\alpha \in A} G_\alpha$ where each G_α is a metric group. Then G is functionally balanced iff each G_α is functionally balanced.*

General Question 3.11. Is a product of functionally balanced T_0 groups functionally balanced? This is unanswered even in the case of a product of two functionally balanced groups. We will show in Section 4 that if the answer is yes for two functionally balanced groups then ‘functional balance’ and ‘balance’ are equivalent for T_0 groups. This actually shows that the answer is positive for arbitrary products of such groups if it is true for two.

Theorem 3.12. *If $G = \operatorname{proj} G_\alpha$, where $(G_\alpha, g_{\beta\alpha})$, $\alpha \in A$, is an inverse limit system of T_0 Lindelöff metric groups of the second category in themselves and if the canonical homomorphisms $g_\alpha : G \to G_\alpha$ are each surjective, then G is balanced iff G is functionally balanced.*

Proof. If the canonical homomorphisms g_α are onto then the connecting maps $g_{\beta\alpha}$ are onto so they are open by Theorem 3.6. Thus the canonical homomorphisms g_α are themselves open. Therefore each G_α is functionally balanced. By Protasov’s theorem [13] on almost metrizable groups, we may conclude that each G_α is balanced, so that $H = \prod_{\alpha \in A} G_\alpha$ is balanced. Since if H is balanced every subgroup of H is balanced it follows that proj G_α is balanced. The converse is clear. $\square$

Remark 3.13. In the following we assume that if $G = \operatorname{proj} G_\alpha$, where $(G_\alpha, g_{\beta\alpha})$, $\alpha, \beta \in A$, is an inverse limit system then the natural homomorphisms $g_\alpha : G \to G_\alpha$ are onto and open. This means that the connecting homomorphisms $g_{\beta\alpha} : G \to G_\alpha$ are open. Note that in [4] the definition of an inverse limit system includes the condition that the connecting maps be open though not necessarily onto. Thus this is a reasonable though slightly stronger condition. The method of proof of Theorem 3.10 yields the following corollaries.

Corollary 3.14. *If $G = \operatorname{proj} G_\alpha$, where $(G_\alpha, g_{\beta\alpha})$, $\alpha \in A$, is an inverse limit system of metric groups such that the the natural homomorphisms $g_\alpha : G \to G_\alpha$ are surjective and open, then G is functionally balanced iff G is balanced.*

Corollary 3.15. *If $G = \operatorname{proj} G_\alpha$, where $(G_\alpha, g_{\beta\alpha})$, $\alpha \in A$, is an inverse limit system of complete Lindelöf metric groups then each natural map $g_\alpha : G \to G_\alpha$ is open if all of the maps g_α are surjective.*

Note 3.16. One more application of Theorem 3.2 can be made. In the case where all the G/H in the proof of the theorem are locally compact or locally countably compact when given the quotient topology then the G/H are locally compact metric groups and the quotient maps are open so that if G is functionally balanced then each G/H is functionally balanced. Thus in this case the coordinate metric groups of the product are balanced. This means by Theorem 3.2 that G is a subgroup of balanced metric groups and so must be balanced. Thus the following is true.

Theorem 3.17. *Let G be a T_0 topological group in which every neighborhood U of the identity contains a closed normal G_δ subgroup H_U. If each of the groups G/H_U is locally compact or locally countably compact then G is balanced iff G is functionally balanced.*

4. Postscript, the work of Protasov and Saryev and the state of the problem

After the topological group workshop was completed a small group consisting of the author, Peter Nickolas, Vladimir Pestov, and Ta Sun Wu met to discuss the present state of the balance vs. functional balance problem for T_0 groups. The first topic discussed was the Protasov–Saryev Theorem 2.6 which played such an important role in the presentation of the workshop material of Section 3. The author could supply a proof of the case (b) $\Rightarrow$ (a), while Pestov could supply a proof of the case (a) $\Rightarrow$ (b) (though neither knew a full proof of the theorem). It should be noted that the theorem appeared in an obscure journal [14] that is not even easily available in Moscow (according to Pestov). Thus the full proof until now has not been available in English and is probably unknown to most researchers working in topological groups. The second topic that was discussed concerned a theorem of Protasov that appeared in [13] in English translation. This theorem (in slightly augmented form) gives a characterization of balance in T_0 groups that reduces the problem of determining whether balance and functional balance are equivalent to a solution of the following problem. Is the product of two functionally balanced groups functionally balanced? The statements and proofs of these two theorems will be presented in this section. It should be noted that Nickolas made the observation that this theorem of Protasov [13] reduced the original problem of 'balance vs. functional balance' to the solution of this new problem.

We begin by quoting a classical theorem appearing in [11, p. 183, Theorem 11].

Theorem 4.1. *Let $(X, \mathcal{V})$ be a uniform space and let d be a pseudometric for X. Then d is uniformly continuous on $X \times X$ relative to the product uniformity iff the set $\{(x, y) \mid d(x, y) < r\}$ is a member of $\mathcal{V}$ for each positive number r.*

Remark 4.2. In the case of a T_0 group G, a pseudometric d is uniformly continuous on $G \times G$ relative to the product left uniform structure iff the set $\{(x, y) \mid d(x, y) < r\}$

is a member of the left uniformity on G for each positive number r. To understand this statement we give the definition of the left uniformity $\mathcal{U}_l$ on $G \times G$. (The right uniformity $\mathcal{U}_r$ is defined similarly.)

Let $\mathcal{U}$ be the neighborhood system at the identity in G.

Definition 4.3. Let $L_U = \{(x, y) \in G \times G \mid x^{-1}y \in U \in \mathcal{U}\}$. Then the left uniformity $\mathcal{U}_l$ is generated by the sets $\{L_U \mid U \in \mathcal{U}\}$ (that is, $\{L_U \mid U \in \mathcal{U}\}$ is a neighborhood base at Δ, the diagonal of $G \times G$).

Thus to say that $\{(x, y) \mid d(x, y) < r\} \in \mathcal{U}_l$ means that there is a set $U \in \mathcal{U}$ such that $L_U \subset \{(x, y) \mid d(x, y) < r\}$. This means that if $x^{-1}y \in U$ then $d(x, y) < r$. Note that $L_U[x] = xU$.

Definition 4.4. A pseudometric d on a T_0 group G is left translation invariant if $d(ax, ay) = d(x, y)$ for all $a \in G$.

Theorem 4.5. *A left uniformly continuous pseudometric d on $G \times G$ is left translation invariant iff $\{(x, y) \mid d(x, y) < r\} = L_U$ for some $U \in \mathcal{U}$.*

Proof. We note that d is left translation invariant iff $d(x, y) = d(e, x^{-1}y)$ for all $x, y \in G$. To see this, note that if $d(x, y) = d(e, x^{-1}y)$ for each $x, y \in G$ then $d(ax, ay) = d(e, (ax)^{-1}ay) = d(e, x^{-1}y) = d(x, y)$. The converse is clear. Now let $U_r = \{(x, y) \mid d(x, y) < r\} = \{(x, y) \mid d(e, x^{-1}y) < r\}$, so that $U_r \in \mathcal{U}_l$ and let $U = U_r[e]$. Then U is a neighborhood of e and $x^{-1}y \in U$ iff $y \in xU = L_U[x]$ iff $(x, y) \in L_U$. Thus $L_U = U_r$, proving the theorem. □

Note 4.6. In a uniform space if $f(x)$ is a uniformly continuous function with respect to the uniformity on the space one can always construct a pseudometric p by the formula $p(x, y) = |f(x) - f(y)|$. In a T_0 topological group there is a standard construction of a left translation invariant pseudometric given by the following theorem proved in [4, 8.2]:

Theorem 4.7. *Let U_n, $n \in \mathbb{N}$, be a sequence of symmetric neighborhoods of e in a topological group G such that $(U_{k+1})^2 \subset U_k$ for $k = 1, 2, 3, \ldots$. Let $H = \bigcap_{k\in\mathbb{N}} U_k$. Then there is a left translation invariant pseudometric σ on G such that*:

(i) *σ is uniformly continuous for the left uniform structure on $G \times G$.*
(ii) *$\sigma(x, y) = 0$ iff $y^{-1}x \in H$.*
(iii) *$\sigma(x, y) \leqslant 2^{-k+2}$ whenever $y^{-1}x \in U_k$.*
(iv) *$\sigma(x, y) \geqslant 2^{-k}$ whenever $y^{-1}x \notin U_k$.*

If, in addition, $xU_kx^{-1} = U_k$, for all x in G and $k = 1, 2, 3, \ldots$, then σ is also right invariant.

Notes 4.8.

(i) The left invariant pseudometric of Theorem 4.7 is bounded and in fact $\sigma(x, y) \leqslant 2$ for all $x, y \in G$. [The left invariant pseudometric is obtained, from a left uniformly continuous function f that is bounded by 1, by the formula $\sigma(x, y) = \sup\{|f(ax) - f(ay)| \mid a \in G\}$.]

(ii) Let U be a fixed symmetric neighborhood of e. Let $U_1 = U$ and let $\{U_k \mid k = 1, 2, \ldots\}$ be a sequence of symmetric neighborhoods of e satisfying $(U_{k+1})^2 \subset U_k$ for $k = 1, 2, \ldots$. Let d_U be the left invariant pseudometric obtained from the proof of the theorem. Let $U(\delta) = \{x \mid d_U(x, e) < \delta\}$ so that $L_{U(\delta)} = \{(x, y) \mid d_U(x, y) < \delta\} = \{(x, y) \mid x^{-1}y \in U(\delta)\}$. Note that by Theorem 4.7(iii), if $x^{-1}y \notin U_2$ we have $d_U(x, y) > 1$. Therefore, $U(1) = \{x \mid d_U(x, e) < 1\} = \{x^{-1}y \mid d_U(x, y) < 1\} \subset U_2 \subset U$. This shows that the collection of neighborhoods of e of the form $\{U(\delta) \mid U$ symmetric neighborhood of $e, d_U(x, e) < \delta\}$ is a base for the neighborhood system at e. In the sequel, if no confusion is possible we will use d to denote the translation invariant pseudometric d_U obtained from U via Theorem 4.7.

We recall Theorem 2.6.

Theorem 2.6 (Protasov and Saryev). *Let G be a T_0 topological group and let $\mathcal{U}$ be a neighborhood base at e consisting of symmetric neighborhoods. Then the following are equivalent*:

(a) *G is functionally balanced.*

(b) *For each $A \subset G$ and each pair $U, W \in \mathcal{U}$ there is a set $V \in \mathcal{U}$ such that $VA \subset AU$ and a set $V' \in \mathcal{U}$ such that $AV' \subset WA$.*

Proof. We will show that the statement,

(a′) In G every left uniformly continuous bounded real valued function is right uniformly continuous, is equivalent to the statement

(b′) For each $U \in \mathcal{U}$ and A in G there is a set $V \in \mathcal{U}$ such that $VA \subset AU$.

The proof that the statements,

(a″) In G every right uniformly continuous bounded real valued function is left uniformly continuous, and

(b″) For each $W \in \mathcal{U}$ and A in G there is $V' \in \mathcal{U}$ such that $AV' \subset WA$, are equivalent,

is similar.

(a′) ⇒ (b′) Let A be a nonempty subset of G, and let $U \in \mathcal{U}$. Choose a left invariant bounded left uniformly continuous pseudometric d on G such that $B_d(e, 1) = \{x \in G \mid d(x, e) < 1\} \subset U$ (see Note 4.8(ii)). Define the real valued function d_A by $d_A(x) = \inf\{d(x, a) \mid a \in A\} = d(x, A)$. Note that d_A is bounded because the pseudometric d is bounded.

Claim. *d_A is left uniformly continuous.*

Proof. Let $x, y \in G$ satisfy $x^{-1}y \in B_d(e, \delta) = \{x \mid d(x, e) < \delta\}$ so that $d(x^{-1}y, e) = d(x, y) < \delta < 1$. From the triangle inequality we have $d(x, A) \leqslant d(x, y) + d(y, A) <$

$\delta + d(y, A)$ and $d(y, A) \leqslant d(y, x) + d(x, A) < \delta + d(x, A)$. Thus $|d(x, A) - d(y, A)| = |d_A(x) - d_A(y)| < \delta$. Since $B_d(e, \delta)$ is a neighborhood of e the claim is proved. □

By our assumption, d_A must be right uniformly continuous. Thus there is a symmetric neighborhood V of the identity with the property that $xy^{-1} \in V$ implies that $|d_A(x) - d_A(y)| < 1$.

Let $x \in VA$. Then $x = va$ for some $v \in V$ and $a \in A$. Since $xa^{-1} = (va)a^{-1} = v \in V$, we conclude that $|d_A(x) - d_A(a)| < 1$. Since $d_A(a) = 0$ we see that $d_A(x) < 1$ means that for some $b \in A$ we have $d(x, b) < 1$ so that by left invariance of d, $d(b^{-1}x, e) < 1$. Thus $b^{-1}x \in B_d(e, 1) \subset U$. Thus $x = b(b^{-1}x) \in AU$ so that $VA \subset AU$.

(b′) ⇒ (a′) Let $\delta > 0$ and let f be a left uniformly continuous bounded real valued function. We may assume that $\|f\|_\infty = 1$ and that $f \geqslant 0$. (If we do not assume f is nonnegative then just prove the theorem for $f^+ = f \vee 0$ and $-f^-$, where $f^- = f \wedge 0$.) Then there is $U \in \mathcal{U}$ such that $x^{-1}y \in U$ implies that $|f(x) - f(y)| < \delta/2$. Let n satisfy $1/n < \delta/4$. Define

$$A_k = \{x\colon 1 - k/n \leqslant f(x) < 1 - (k-1)/n\}.$$

Note that if $x, y \in A_k$, then $|f(x) - f(y)| \leqslant |f(x) - (1 - (k-1)/n)| + |(1 - (k-1)/n) - f(y)| < 1/n + 1/n < \delta/2$. By hypothesis, there is $V_k \in \mathcal{U}$ such that $V_k A_k \subset A_k U$. Note that if $yx^{-1} \in V_k$ and $x \in A_k$ then $y \in V_k x \subset V_k A_k \subset A_k U$ so that $y \in zU$ for some $z \in A_k$. Therefore $|f(x) - f(y)| \leqslant |f(x) - f(z)| + |f(z) - f(y)| < \delta/2 + \delta/2 = \delta$. (Thus f is right uniformly continuous on A_k.) Since this argument can be applied to each A_k, there is a finite collection of neighborhoods V_k, $k = 1, \ldots, n$, of the identity such that $x \in A_k$ and $yx^{-1} \in V_k$ implies that $|f(x) - f(y)| < \delta$. Let now $V = \bigcap_{i \leqslant n} V_i$. Then $V \in \mathcal{U}$.

Note that if x is in G then $x \in A_k$ for some $k \leqslant n$. This means that if $yx^{-1} \in V \subset V_k$, then $|f(x) - f(y)| < \delta$, so that f is right uniformly continuous on all of G. □

Notes 4.10.

(i) The proof of (b′) ⇒ (a′) was suggested to the author by the following classical theorem from measure theory.

Theorem. *A bounded measurable real valued function can be approximated from below* (*and also from above*) *by an increasing sequence of simple functions.*

The proof applied to the real line actually shows that the approximation is uniform on the measurable sets. This fact was always pointed out to the students in his class by the author after Egorov's theorem was proved.

(ii) The proof of (b′) ⇒ (a′) cannot be used to show that if (b′) holds then f is strongly functionally balanced. The reason for this is that in general there would be an infinite number of pairs A_k, V_k and there is no reason to expect that $\bigcap V_i$ is a neighborhood of e. This led to Question 2.7.

The next theorem basically appeared in Protasov's paper [13]. In his paper he proved that if for a T_0 group G, the group $G \times G$ is functionally balanced, then G is balanced.

We note that the product left uniform structure on $G \times G$ is the same as the left uniform structure on $G \times G$ (see [1, III.3.1]. His statement can be slightly augmented as follows:

Theorem 4.11. *Let G be a T_0 topological group. Then the following are equivalent*:

(a) *$G \times G$ is functionally balanced.*

(b) *Each left translation invariant pseudometric continuous for the product left uniform structure on $G \times G$ is continuous for the product right uniform structure on $G \times G$.*

(c) *G is balanced.*

Proof. (a) $\Rightarrow$ (b) is clear.

(b) $\Rightarrow$ (c) Let U be a symmetric neighborhood of e and let d be the left invariant pseudometric that is continuous for the left uniform structure on $G \times G$ that Theorem 4.4 associates with U. Let $B(e,1) = \{x \mid d(e,x) < 1\} \subset U$. Since d is continuous for the right structure on $G \times G$, there is a neighborhood V of e such that $yx^{-1} \in V$ implies that $d(x,y) < 1$. Let x be fixed. If $v \in V$, we have $(vx)x^{-1} = v \in V$ so that $d(vx,x) = d(x,vx) < 1$. Therefore $(vx)^{-1}x \in U$. Since U is symmetric, $x^{-1}vx \in U$, so that $vx \in xU$. Since v was arbitrary in V we have $Vx \subset xU$. Since x was arbitrary in G we have $Vx \subset xU$, for all x in G. Therefore $V \subset xUx^{-1}$ for all x in G, and G is balanced.

(c) $\Rightarrow$ (a) Since G is balanced, Theorem 3.7 tells us that $G \times G$ is balanced. It is now easy to check that each balanced group is functionally balanced. $\square$

Note 4.12. The method of proof of (b) $\Rightarrow$ (c) suggests the next theorem. This theorem essentially answers a question in an earlier version of this paper that was distributed at the topology workshop.

Theorem 4.13. *Let d be a left translation invariant pseudometric continuous for the product left uniform structure on $G \times G$. If d is continuous for the right uniform structure on $G \times G$, then the unit ball $U_1 = \{x \mid d(e,x) < 1\}$ contains a closed normal G_δ subgroup H.*

Proof. First note that U_1 is a neighborhood of e. By the proof of Theorem 4.11 there is a neighborhood V of e such that $V \subset xU_1x^{-1}$ for each x in G. Therefore $V \subset W = \bigcap\{xU_1x^{-1} \mid x \in G\}$ and W is a balanced neighborhood of e. For each $k = 1, 2, 3, \ldots$, let $U_k = \{x \mid d(e,x) < 1/2^k\}$.

Claim. *For each k, $(U_{k+1})^2 \subset U_k$.*

Proof. If $x \in (U_{k+1})^2$ there is $y \in U_{k+1}$ such that $x \in yU_{k+1}$. Since $y^{-1}x \in U_{k+1}$, the left translation invariance of d shows that $d(x,y) = d(y^{-1}x, e) < 1/2^{k+1}$. Therefore, the triangle inequality shows that $d(e,x) \leqslant d(e,y) + d(y,x) < 1/2^{k+1} + 1/2^{k+1} = 1/2^k$ so that $x \in (U_{k+1})^2$ proving the claim. $\square$

Since d is continuous for the right uniform structure a similar argument shows that for each k, there are open neighborhoods V_k, W_k of the identity e satisfying $V_k \subset W_k =$

$\bigcap\{xU_kx^{-1} \mid x \in G\}$, so that each W_k is balanced. Note that if $z \in (W_{k+1})^2$, then $z = z_1z_2$ where $z_1, z_2 \in W_{k+1}$. Thus for fixed x, $z_1 = xux^{-1}$ and $z_2 = xu'x^{-1}$, where $u, u' \in U_{k+1}$. Thus $z = z_1z_2 = xux^{-1}xu'x^{-1} = xuu'x^{-1} \in x(U_{k+1})^2x^{-1} \subset xU_kx^{-1}$. Therefore $z \in W_k = \bigcap\{xU_kx^{-1} \mid x \in G\}$ so that $(W_{k+1})^2 \subset W_k$. Since the W_k are a collection of balanced neighborhoods Theorem 4.7 tells us that the pseudometric d_1 associated with the neighborhoods $\{W_k\}$ is not only left invariant but also right invariant. Furthermore the set $H = \bigcap\{W_k \mid k = 1, 2, 3, \ldots\}$ is a closed G_δ normal subgroup of G. It is clear that $H \subset U_1$. □

Theorem 4.14. *The following are equivalent for T_0 groups*:

(a) *Each product of two functionally balanced groups is functionally balanced.*

(b) *If G is functionally balanced then $G \times G$ is functionally balanced.*

(c) *Each functionally balanced group is balanced.*

Proof. (a) $\Rightarrow$ (b) is clear.

(b) $\Rightarrow$ (c) This is Theorem 4.11.

(c) $\Rightarrow$ (a) If each functionally balanced group is balanced then the product of any two of them is balanced and therefore functionally balanced. □

Note 4.15. The implication (b) $\Rightarrow$ (c) of Theorem 4.14 was noted by Nickolas during the miniconference held after the workshop when the author mentioned Protasov's result. At the topological groups workshop, Pestov had outlined a much longer procedure that would show that if the product of any two functionally balanced groups was functionally balanced then an arbitrary product of functionally balanced groups would also be functionally balanced. This observation of Nickolas gives a quick proof of the last fact. In fact, if the product of any two functionally balanced groups are functionally balanced then each functionally balanced group is balanced so an arbitrary product of functionally balanced groups is balanced (by Theorem 2.8) and hence functionally balanced. In any event this observation shows that the question of 'balance' vs. 'functional balance' boils down to answering the following question.

Question 4.16. Is the product of any functionally balanced T_0 group G with itself functionally balanced?

If the answer is yes, the functionally balanced groups are balanced. If the answer is no, then $G \times G$ is not balanced and so G cannot be balanced. Thus one should look for an example of a functionally balanced group G such that $G \times G$ is not functionally balanced. A final question can be added to the ones already posed on this problem.

Question 4.17. If every left uniformly continuous bounded real valued function on a group G is right uniformly continuous, then is each left invariant pseudometric continuous for the left uniform structure on $G \times G$ also continuous for the right structure on $G \times G$?

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